Lorentz-Invariant Interpretation of Noncommutative Space-Time - global version.

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Abstract

The global version of the quantum symmetry defined by Chaichian et al (hep-th/0408069) is constructed.

*supported by UŁ grant no. 690

In the very recent interesting paper Chaichian and al. [1] proposed a new interpretation of the symmetry of noncommutative space-time defined by the commutation relations:

$$[x_\mu,\ x_\nu] = i\Theta_{\mu\nu}, \tag{1}$$

where $\Theta_{\mu\nu}$ is a constant antisymmetric matrix. According to the standard wisdom the relations (1) break the Lorentz symmetry down to the stability subgroup of $\Theta_{\mu\nu}$. In spite of that all fundamental issues of the noncommutative quantum field theory (NCQFT) are discussed in fully covariant approach using the representations of Poincare group. The reason for that might be that NCQFT emerges as specific limit of fully symmetric theory.

On the other hand one can pose the question whether the noncommutative space-time admits as large symmetry as its commutative counterpart provided the symmetry is understood in the wider sense (for example, as a symmetry in the sense of the quantum group theory). This is important because if one tries to base the theory on the stability subgroup of $\Theta_{\mu\nu}$ one is faced at once with deep problems [2] (for example, why the multiplets of stability subgroup are organized in such a way as to form the complete multiplets of the whole group). The problem can be posed in quite general terms. Given a theory based on some symmetry group broken explicitly to its subgroup it is usually not sufficient to use the properties of this subgroup. Some questions can be answered only within the framework of the initial symmetry in spite of the fact that it is broken, i.e. it is formally no longer a symmetry.

The solution to this dilemma might be as follows. Assume the original symmetry group is broken by some additional conditions imposed (like Poincare group being broken by a specific choice of $\Theta_{\mu\nu}$). Then it appears that some properties of the system can be explained in terms of the residual symmetry while in order to understand other properties one has to appeal to the initial symmetry. Assume further that we have found a quantum symmetry as large as the initial classical one. It provides a deformation of the classical symmetry, the parameter of deformation being determined by the strength of symmetry breaking. Quantum symmetry is a more general notion and, therefore, one can expect its consequences are weaker. Ideally, we can hope that the quantum symmetry implies some conclusions of initial classical symmetry (e.g. the classification of multiplets) survive while other (modified by symmetry breaking) do not.

It has been shown in ref. [1] that the quantum symmetry of noncommutative space-time defined by eq. (1) is a twisted Poincare algebra, the twist element being an abelian twist [3]:

$$F = exp(\frac{i}{2}\Theta^{\mu\nu}P_\mu \otimes P_\nu) \tag{2}$$

Twisting the Poincare algebra provides an infinitesimal form of quantum symmetry of noncommutative space-time. The first step in checking whether the above sketched scenario works in NCQFT is to analyse the mathematical structure of the quantum symmetry found in ref. [1]. In the present note we give the global version of twisted Poincare symmetry of Chaichian et al.

Our starting point is the matrix form of Poincare transformations. Namely, we consider 5×5 matrices $T^a_b,\ a,\ b = 0,\ 1,\ ...,\ 4,$ of the form

$$T = \begin{bmatrix} \Lambda^\mu{}_\nu & | & a^\mu \\ ---- & | & ---- \\ 0 & | & 1 \end{bmatrix}; \tag{3}$$

here $\mu,\quad \nu = 0,\ ...,\ 3,\ \Lambda^\mu{}_\nu$ is Lorentz matrix while a^μ denotes translation. The composition law for Poincare group can be now written as

$$\Delta T^a{}_b = T^a{}_c \otimes T^c{}_b \tag{4}$$

We take eq. (4) as the definition of coproduct of our quantum Poincare group. In order to find the algebraic structure one can use the FRT relation [4]

$$RTT = TTR \tag{5}$$

where R is the universal R-matrix in the representation determined by T. Now, the R-matrix for a given twist F of classical group reads [3]

$$R = F_{21}F^{-1} \tag{6}$$

In our case F is given by eq. (2) while P_μ can be computed from (3). The calculations are greatly simplified by the fact that P_μ are nilpotent matrices.

Skipping the details we present the final result. The quantum Poincare group dual to the algebra considered in ref. [1] is defined by the following

relations

$$
\begin{aligned}
&\Delta\Lambda^{\mu}{}_{\nu} = \Lambda^{\mu}{}_{\alpha} \otimes \Lambda^{\alpha}{}_{\nu} \\
&\Delta a^{\mu} = \Lambda^{\mu}{}_{\alpha} \otimes a^{\alpha} + a^{\mu} \otimes 1 \\
&\varepsilon(\Lambda^{\mu}{}_{\nu}) = \delta^{\mu}{}_{\nu} \\
&\varepsilon(a^{\mu}) = 0 \\
&S(\Lambda^{\mu}{}_{\nu}) = \Lambda_{\nu}{}^{\mu} \\
&S(a^{\mu}) = -\Lambda_{\alpha}^{\ \mu} a^{\alpha} \\
&[\Lambda^{\mu}{}_{\nu}, \cdot] = 0 \\
&[a^{\mu},\ a^{\nu}] = -i\Theta^{\rho\sigma}(\Lambda^{\mu}{}_{\rho}\Lambda^{\nu}{}_{\sigma} - \delta^{\mu}{}_{\rho}\delta^{\nu}{}_{\sigma})
\end{aligned}
\tag{7}
$$

and $*$-involution is defined by

$$
\begin{aligned}
&(a^{\mu})^{*} = a^{\mu} \\
&(\Lambda^{\mu}{}_{\nu})^{*} = \Lambda^{\mu}{}_{\nu}
\end{aligned}
\tag{8}
$$

One can check by straightforward calculations that the above structure is consistent and defines $*$-Hopf algebra. Note that, contrary to the κ-Poincare case [5], the translations do not form a subalgebra. In spite of that, one can define the action of our Poincare group on quantum space-time defined by eqs. (1) by

$$
x^{\mu} \rightarrow \Lambda^{\mu}{}_{\nu} \otimes x^{\nu} + a^{\mu} \otimes I \tag{9}
$$

It is easy to check that this (co-) action is well defined and consistent with the commutation rules (1).

Having defined the deformed Poincare group and its action on quantum Minkowski space (eqs. (7)-(9)) one can follow the standard rules of quantum group theory [6] to find the representations, to classify the differential calculi on quantum group and representation space, etc. This should provide the tools for constructing a quantum group invariant dynamics and to see whether one can explain the structure of NCQFT without appealing to larger structures.

Note added

After submitting this paper to hep archive we have learned from R. Oeckl that our result is contained in his paper in Nucl. Phys.B 581, (2000), 559.

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